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BILINEAR APPROXIMATION OF KERNELS OF VOLTERRA EQUATION OF THE SECOND KIND

This article examine algorithm to approximate function of two variables in the form of a bilinear series and also their application to the solution of Volterra integral equations of the second kind by using method of separate kernels.

Key words: *approximation, algorithm, a separate kernel, the integral equation.*

Recently there has been a significant expansion of application area of Volterra equation of the second kind

$$y(x)-\int_a^x K(x,s)\,y(s)\,ds = f(x), \tag{1}$$

where $y(s)$ — desired function, $x,s\in[a,b]$, kernel $K(x,s)$ and right side $f(x)$ — specified.

The main problems that make up the range of applications of this class of equations are problems of modeling dynamic objects and often real-time modeling problems. The solution of similar problems requires high speed and accuracy of computer tools.

Replacement of integral with quadrature formula that underlies the quadrature method for solving equations [1] represents the most direct and in most practical cases the most effective way to prepare Volterra equation for computer implementation.

Under the condition that kernel of integral equation is separable

$$K(x,s)=\sum_{i=1}^{l}\alpha_i(x)\,\beta_i(s) \tag{2}$$

following method can be used. Let's write equation (1) for separable kernel

$$y(x)-\sum_{i=1}^{l}\alpha_i(x)\int_a^x\beta_i(s)\,y(s)\,ds = f(x). \tag{3}$$

Converting equation (2) into discrete form results in

$$y(x_p)-\sum_{i=1}^{l}\alpha_i(x_p)\int_a^x\beta_i(s)\,y(s)\,ds = f(x_p), \tag{4}$$

$p=\overline{1,n}$ which will allow to use quadrature formula of trapezoid with constant step h for solving integral equation and obtain the following recurrent expression:

$$\begin{cases} y_1 = f_1, \\ y_k = \dfrac{f_k + h\sum_{i=1}^{l}\alpha_i(x_k)\sum_{p=1}^{k-1}\beta_i(x_p)A_p y_p}{1 - h/2\sum_{i=1}^{l}\alpha_i(x_k)\beta_i(x_k)}, \end{cases} \tag{5}$$

where $k=\overline{2,n}$, $A_p = \begin{cases} 0,5\ if\ \ p=1 \\ 1\ if\ \ p>1 \end{cases}$. Expression (4) allows to keep the number of calculation on every step the same, because for calculating each of l sums $\sum_{p=1}^{k-1}\beta_i(x_p)A_p y_p$ results from previous step $\sum_{p=1}^{k-1}\beta_i(x_j)A_p y_p = \sum_{j=1}^{k-2}\beta_i(x_p)A_p y_p + \beta_i(x_{p-1})A_{p-1}y_{p-1}$, $k=\overline{3,n}$ can be used and therefore it's appropriate to use feature of separable kernels for solving Volterra equation. However this feature doesn't occur often therefore it is necessary to have an effective method for representing any kernel in separable form. One of possible approaches suggested further.

Let's consider the problem of constructing a separable kernel as obtaining sequentially members of bilinear sum (2) [2] in square $[a \le x, s \le b]$ with the condition of a minimum of functional

$$\begin{aligned} \Phi = \int_a^b\int_a^b\Big[K(x,s) - \sum_{i=1}^{l}\alpha_i(x)\beta_i(s)\Big]^2 dxds + \\ +n\int_a^b\int_a^b\Big[K_x^{'}(x,s) - \sum_{i=1}^{l}\alpha_{i'}(x)\beta_i(s)\Big]^2 dxds + \\ +m\int_a^b\int_a^b\Big[K_s^{'}(x,s) - \sum_{i=1}^{l}\alpha_i(x)\beta_{i'}(s)\Big]^2 dxds \to min, \end{aligned} \tag{6}$$

where n and m are weight coefficients at summands which consider first derivatives $K(x,s)$ with respect to x and s in accordance and can be set based on function nature or as 0.

Equating the left side of functional (6) to zero (ideal case approximation) allows to obtain through equivalent transformations an expression for finding desired $\alpha_i(x)$ as a differential equation

$$\begin{aligned} \alpha_i^{''}(x)n\int_a^b\beta_i(s)ds - \alpha_i(x)\Big(m\int_a^b\beta_{i'}(s)ds + \int_a^b\beta_i(s)ds\Big) = \\ = \int_a^b\Big(K(x,s) + mK_x^{'}(x,s) + nK_{xx}^{''}(x,s)\Big)ds \end{aligned} \tag{7}$$

with boundary values

$$\alpha^{'}(a) = \frac{\int_a^b K_x^{'}(a,s)ds}{\int_a^b\beta_i(s)ds}, \alpha^{'}(b) = \frac{\int_a^b K_s^{'}(b,s)ds}{\int_a^b\beta_i(s)ds} \tag{8}$$

and similar equation for $\beta_i(s)$

$$\beta_i''(s)m\int_a^b \alpha_i(x)ds - \beta_i(s)\left(m\int_a^b \alpha_{i'}(x)dx + \int_a^b \alpha_i(x)dx\right) = \\ = \int_a^b \left(K(x,s) + nK_s'(x,s) + mK_{ss}''(x,s)\right)dx \tag{9}$$

with boundary values

$$\beta_i'(a) = \frac{\int_a^b K_x'(a,s)ds}{\int_a^b \beta_i(s)ds}, \ \beta_i'(b) = \frac{\int_a^b K_s'(b,s)ds}{\int_a^b \beta_i(s)ds}. \tag{10}$$

The solution of equations (7) and (9) gives the desired functions $\alpha_i^{(j)}(x)$ and $\beta_i^{(j)}(s)$. It's necessary to take into account that this procedure requires multiple execution of differentiation operation that in case of numerical calculations can lead to large errors and often to the inability of obtaining the result with required accuracy.

To simplify the calculation formulas and to increase calculation stability expressions $\alpha_i''(x)n\int_a^b \beta_i(s)ds$, $nK_{xx}''(x,s)$ for (7) and $\beta_i''(s)m\int_a^b \alpha_i(x)ds$, $mK_{ss}''(x,s)$ for (9) can be neglected which is justified for small values of n and m. Using obtained expressions in turn

$$\alpha_i^{(j)}(x) = \frac{\int_a^b K_i(x,s)\beta_i^{(j)}(s) + mK_x'(x,s)\beta_i^{(j)\prime}(s)ds}{\int_a^b \left(\beta_i^{(j)}(s) + \beta_i^{(j)\prime}(s)\right)^2 ds}, \tag{11}$$

$$\beta_i^{(j)}(s) = \frac{\int_a^b K_i(x,s)\alpha_i^{(j)}(x) + nK_s'(x,s)\alpha_i^{(j)\prime}(x)dx}{\int_a^b \left(\alpha_i^{(j)}(x) + \alpha_i^{(j)\prime}(x)\right)^2 dx} \tag{12}$$

for forming $\alpha_i(x)\beta_i(s)$, where $K_1(x,s) = K(x,s)$, $\beta_i^{(0)}(s)$ specified priori and replacement $i = i+1$ and $K_i(x,s) = K_{i-1}(x,s) - \alpha_{i-1}(x)\beta_{i-1}(s)$ occurs under the condition

$$\begin{cases} \left\|\alpha_i^{(j)}(x) - \alpha_i^{(j-1)}(x)\right\| \le \varepsilon, \\ \left\|\beta_i^{(j)}(s) - \beta_i^{(j-1)}(s)\right\| \le \varepsilon, \end{cases} \tag{13}$$

where ε — specified accuracy index for calculating functions $\alpha_i(x)$ and $\beta_i(s)$. New terms of series are made until condition is fulfilled

$$\left\|K(x,s) - \sum_{i=1}^{l} \alpha_i(x)\beta_i(s)\right\| \le \varepsilon_{apr}, \tag{14}$$

where ε_{apr} specified accuracy index for approximation. This way we get series (2) which approximates kernel $K(x,s)$ with required accuracy.

Using given algorithm developed programs which resolved following examples.

Example 1. Solving the Volterra equation of the second kind

$$y(x)-\int_a^x e^{-x-s}y(s)\,ds=e^{-x}, \tag{15}$$

with known solution $y(x)=1$ in advance, fig. 1 shows function $K(x,s)=e^{-x-s}$

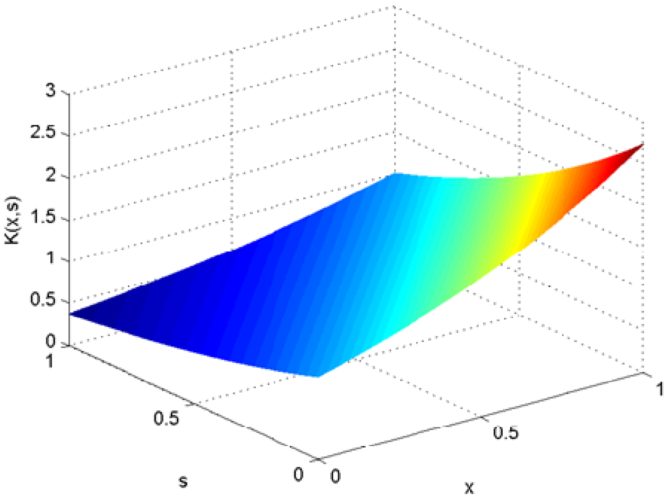

***Fig. 1.** Function* $K(x,s)=e^{-x-s}$

Table 1

Solving method	**Quadrature method of trapezoid**	**Quadrature method of trapezoid considering separable kernel**
Terms of solution	$0<x,s<1$, $h=0.1^3$	$0<x,s<1$, $h=0.1^3$, $n=m=0$, $\varepsilon=0.1^4$, $\varepsilon_{apr}=0.1^5$
Elapsed time	0.006681 s	0.005262 s
The maximum deviation obtained from the exact solution	$8.3494*10^{-8}$	$8.3493*10^{-8}$

Example 2. Solving the Volterra equation of the second kind

$$y(x)-\int_a^x x\sin\left(xs^3\right)^2 y(s)\,ds = x^2-\frac{1}{tg\left(x^4\right)}, \tag{16}$$

fig. 2 shows function $K(x,s)=x\sin\left(xs^3\right)^2$.

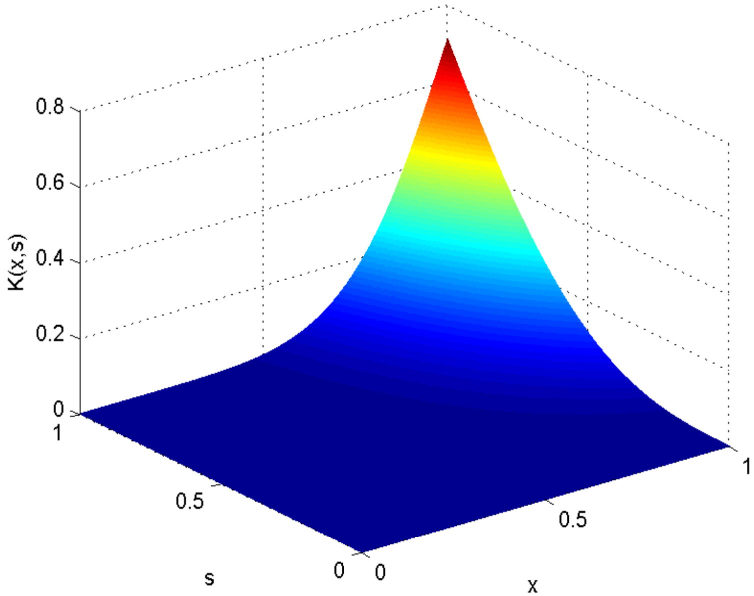

Fig. 2. *Function* $K(x,s)=x\sin\left(xs^3\right)^2$

Table 2 indicates the results of solving equation (16) and fig. 3 shows solution.

Table 2

Solving method	**Quadrature method of trapezoid**	**Quadrature method of trapezoid with separable kernel construction**	
Terms of solution	$0<x,s<1$, $h=0.1^3$	$0<x,s<1$, $h=0.1^3$, $n=m=0$, $\varepsilon=0.1^4$, $\varepsilon_{apr}=0.1^5$	$0<x,s<1$, $h=0.1^3$, $n=m=0.05$, $\varepsilon=0.1^4$, $\varepsilon_{apr}=0.1^5$
Elapsed time	0.016696 s	0.009243 s	0.009149 s

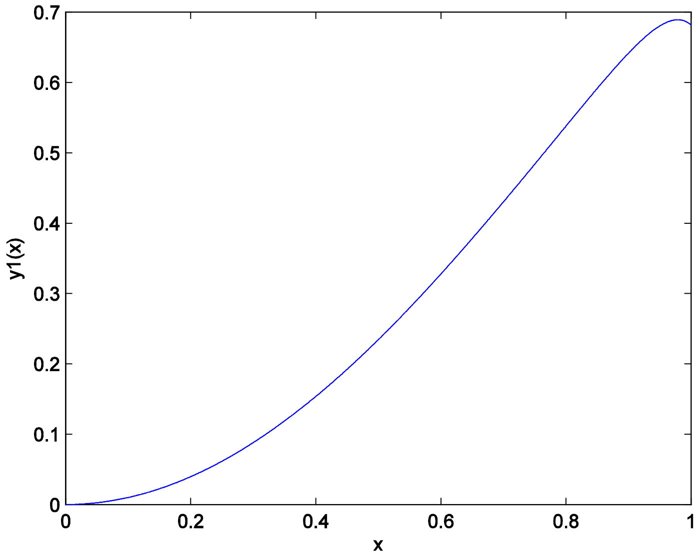

Fig. 3. *Solution* $y(s)$

Use of numerical algorithms for approximation kernels of integral equations, as functions of two variables allows to improve performance and extend the application of the method of separable kernels.

Referenses

1. Manzhyrov A. V. Справочник по интегральным уравнениям: Методы решения / A. V. Manzhyrov, A. D. Polyanyn. — М. : Факториал Пресс, 2000. — 384 p.
2. Верлань Д. А. Ітераційні алгоритми апроксимації функції двох змінних / Д. А. Верлань // Зб. наук праць. Математичне та комп'ютерне моделювання. Серія: Технічні науки. — Кам'янець-Подільський : Кам'янець-Подільський національний університет імені Івана Огієнка, 2009. — Вип. 2. — С. 24–32.

Розглядається алгоритм апроксимації функцій двох змінний у вигляді білінійної ряду, а також їх застосування для розв'язання інтегральних рівнянь Вольтерри II-го роду методом вироджених ядер.

Ключові слова: *апроксимація, алгоритм, вироджене ядро, інтегральне рівняння.*

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